

# A Diffusive System Driven by a Battery or by a Smoothly Varying Field

T. Bodineau · B. Derrida · J.L. Lebowitz

Received: 15 March 2010 / Accepted: 16 June 2010 / Published online: 13 July 2010  
© Springer Science+Business Media, LLC 2010

**Abstract** We consider the steady state of a one dimensional diffusive system, such as the symmetric simple exclusion process (SSEP) on a ring, driven by a battery at the origin or by a smoothly varying field along the ring. The battery appears as the limiting case of a smoothly varying field, when the field becomes a delta function at the origin. We find that in the scaling limit the long range pair correlation functions of the system driven by a battery are very different from the ones known in the steady state of the SSEP maintained out of equilibrium by contact with two reservoirs, even when the steady state density profiles are identical in both models.

**Keywords** Non-equilibrium · Long range correlations · Canonical measure · Fluctuating hydrodynamics

## 1 Introduction

There are notoriously few fully analyzable models of non trivial (interacting) current carrying systems [22, 26] in non-equilibrium steady states (NESS). The few exceptions are almost all one dimensional lattice gases evolving according to stochastic jump processes and interacting via exclusions, in contact with particle reservoirs at different densities. Among the simplest of such models is one of particles on a lattice of  $N$  sites ( $i = 1, \dots, N$ ) in which

---

T. Bodineau (✉)  
Département de Mathématiques et Applications, Ecole Normale Supérieure, CNRS-UMR 8553,  
75230 Paris Cedex 05, France  
e-mail: [bodineau@ens.fr](mailto:bodineau@ens.fr)

B. Derrida  
Laboratoire de Physique Statistique, Ecole Normale Supérieure, UPMC Paris 6, Université Paris  
Diderot, CNRS, 24 rue Lhomond, 75231 Paris Cedex 05, France

J.L. Lebowitz  
Departments of Mathematics and Physics, Rutgers University, 110 Frelinghuysen Road, Piscataway,  
NJ 08854, USA

the bulk dynamics are given by the symmetric simple exclusion process (SSEP) and sites 1 and  $N$  are in contact with particle reservoirs. In the case of reservoirs at equal densities the stationary state satisfies detailed balance. For reservoirs at unequal densities, say  $\rho_a$  on the left and  $\rho_b$  on the right ( $\rho_a > \rho_b$ ), the matrix method gives a full microscopic description of the NESS including the explicit form of the correlation functions  $\rho_k(i_1, \dots, i_k; \rho_a, \rho_b, N)$  [15] and allows one to calculate the large deviation function (LDF) in the hydrodynamical scaling limit,  $N \rightarrow \infty, i/N \rightarrow x \in [0, 1]$  [1, 2, 13]. One can also obtain, in this scaling limit (with diffusively scaled time), the time evolution of the system in a non stationary state via a diffusion equation for typical density profiles as well as the LDF for time evolving densities and currents [1, 3, 5, 6, 11].

In attempts to further extend our understanding of such non-equilibrium systems, we investigate here the stationary state of a system of  $M$  particles on a ring of  $N$  sites with exclusion. The jump rates to the right and left (counter clockwise, clockwise) for a particle at site  $i$  are  $p_i$  and  $q_i, i = 1, \dots, N$ . We shall be interested, as usual, in the case  $N \gg 1, M/N = \bar{\rho}$ . We consider two types of situations:

1. The “battery” case: the system evolves according to the SSEP except that there is a “battery” at bond  $(N, 1)$ , i.e.  $p_i = 1$  for  $i \neq N, q_i = 1$  for  $i \neq 1$  and  $p_N = p, q_1 = q$  with  $p$  and  $q$  independent of  $N$ .
2. The smooth asymmetric case (WASEP)  $p_i = 1 + \frac{1}{N}E(i/N)$  and  $q_i = 1 - \frac{1}{N}E(i/N)$  with  $E$  a smooth field of period 1.

(Combination of both cases can also be treated.)

In the first case, the microscopic “battery” at  $(N, 1)$  induces a NESS with a particle current of magnitude proportional to  $1/N$  and a concomitant linear density profile. This looks very similar to what happens in the open system and would suggest that the measures for the reservoirs and battery driven systems would be similar when  $N \gg 1$ , in the same spirit as the difference between “canonical” and “grand-canonical” nature of the two systems disappears in equilibrium, when  $p = q$  and  $M/N \rightarrow \rho_a = \rho_b$ . This is indeed the case to the leading order in  $N$ . Local equilibrium holds in both models and implies that both steady state statistics are given locally by a product Bernoulli measure. Here, we will focus on the corrections to this local Bernoulli measure and on the long range correlations. We will see that the two NESS measures differ much more than the corresponding equilibrium ones. The complicated interplay between the non-equilibrium long-range correlations, the canonical constraint and the driving mechanism leads to two-point correlation functions with different structure for the reservoir and battery driven systems. In particular, the long range correlations are much more singular for the battery model than those in the open system. A variant of this model with a macroscopic battery was introduced in [19] and a related dynamics with colliding hard spheres was also considered in [9].

In the second case, the macroscopic stationary density profile  $\bar{\rho}(x)$  then satisfies the general equation for driven diffusive systems

$$\partial_x \bar{\rho}(x) - 2E(x)\bar{\rho}(x)(1 - \bar{\rho}(x)) = -\mathcal{J}, \tag{1.1}$$

where  $\mathcal{J}$  is the stationary current which is independent of  $x$ . Equation (1.1) corresponds to the stationary solution of the diffusion equation with a drift

$$\partial_t \rho = \partial_x^2 \rho - \partial_x (2E(x)\rho(1 - \rho)). \tag{1.2}$$

The first case can be thought of as a very singular limit of the second one when  $E(x)$  becomes a  $\delta$ -function localized at the origin.

We will also consider the Zero Range Process (ZRP) driven by a “battery”. This microscopic dynamics is simpler than the SSEP driven by a battery, as the NESS of the ZRP on the ring is a product measure constrained to having a fixed number of particles. There are no long range correlations in the non-equilibrium measure of the ZRP with reservoirs [23] and the long range correlations of the ZRP driven by a battery are similar to those found for an equilibrium system with the constraint that the total particle number is fixed.

The outline of the rest of the paper is as follows. In Sect. 2, we define the microscopic models. By using a macroscopic approach, the two-point correlation functions are computed for the battery model in Sect. 3 and for a slowly varying field in Sect. 4. These results are then compared, in Sect. 5, to the two-point correlation functions of an open system in contact with reservoirs. Finally the invariant measure of the ZRP driven by a battery is computed in Sect. 6. Some technical details concerning the solution of the macroscopic equations are given in the appendices.

## 2 The Models

### 2.1 The Battery Model

We consider the SSEP on the ring  $\{1, N\}$  containing  $M$  particles with jump rates 1 to the left and to the right except at the bond  $(N, 1)$  where the jump rate from  $N$  to 1 is  $p$  and from 1 to  $N$  is  $q$ . These modified asymmetric rates act as a battery which forces a current of particles through the system. We assume that  $p > 0$  and  $q > 0$ .

In the absence of the battery ( $p = q = 1$ ), the stationary state is one in which all the  $\binom{N}{M}$  configurations have equal weight. For  $N \rightarrow \infty$ ,  $M = \bar{\rho}N$ , this corresponds locally to a product Bernoulli measure with density  $\bar{\rho}$ . Denoting by  $v_i^{\bar{\rho}}$  the Bernoulli measure at site  $i$  with density  $\bar{\rho}$ , the product measure  $\bigotimes_{i \in \mathbb{Z}} v_i^{\bar{\rho}}$  is invariant wrt the SSEP dynamics for any constant density  $\bar{\rho}$ .

For the SSEP on  $\mathbb{Z}$  with a battery at the bond  $(0, 1)$ , the invariant measure remains a product but the density is discontinuous across the battery. The invariant measures on  $\mathbb{Z}$  are

$$v^{\rho^-, \rho^+} = \bigotimes_{i \leq 0} v_i^{\rho^-} \bigotimes_{i \geq 1} v_i^{\rho^+}, \tag{2.1}$$

where the densities  $\rho^-, \rho^+$  (at sites 0 and 1) satisfy the equation

$$p\rho^-(1 - \rho^+) = q\rho^+(1 - \rho^-). \tag{2.2}$$

These measures satisfy detailed balance and there is no current flowing in the system.

On the ring when  $p \neq q$ , the steady state measure is not known. Taking averages with respect to the time evolving measure  $\mu_{N,\tau}$ , we obtain

$$\begin{aligned} i \neq 1, N, \quad \partial_\tau \mu_{N,\tau}(\eta_i) &= \mu_{N,\tau}(\eta_{i+1}) + \mu_{N,\tau}(\eta_{i-1}) - 2\mu_{N,\tau}(\eta_i), \\ \partial_\tau \mu_{N,\tau}(\eta_1) &= \mu_{N,\tau}(\eta_2) - \mu_{N,\tau}(\eta_1) + p\mu_{N,\tau}(\eta_N(1 - \eta_1)) \\ &\quad - q\mu_{N,\tau}(\eta_1(1 - \eta_N)), \\ \partial_\tau \mu_{N,\tau}(\eta_N) &= \mu_{N,\tau}(\eta_{N-1}) - \mu_{N,\tau}(\eta_N) - p\mu_{N,\tau}(\eta_N(1 - \eta_1)) \\ &\quad + q\mu_{N,\tau}(\eta_1(1 - \eta_N)). \end{aligned} \tag{2.3}$$

It follows immediately that in the stationary state, the average density profile is linear

$$\langle \eta_i \rangle = \langle \eta_1 \rangle + \frac{i - 1}{N - 1} (\langle \eta_N \rangle - \langle \eta_1 \rangle), \tag{2.4}$$

where  $\langle \cdot \rangle$  stands for the stationary measure and the mean density  $\bar{\rho} = \frac{M}{N}$  satisfies

$$\bar{\rho} = \frac{1}{2} (\langle \eta_N \rangle + \langle \eta_1 \rangle). \tag{2.5}$$

There is also a stationary average current  $\hat{q}$  of order  $1/N$

$$\hat{q} = \frac{1}{N} (\langle \eta_1 \rangle - \langle \eta_N \rangle), \tag{2.6}$$

and one has the identity

$$p \langle \eta_N (1 - \eta_1) \rangle - q \langle \eta_1 (1 - \eta_N) \rangle = \hat{q}. \tag{2.7}$$

For finite  $N$ , the relations (2.4) and (2.7) do not allow the determination of the average profile  $\langle \eta_i \rangle$  as (2.7) involves the correlation between sites  $N$  and  $1$ . If one assumes however that local equilibrium holds for large  $N$  then the system will behave at the battery as if it was described by the measure (2.1). Thus one expects that for large  $N$

$$\langle \eta_N \eta_1 \rangle \approx \langle \eta_1 \rangle \langle \eta_N \rangle = \rho^+ \rho^-, \tag{2.8}$$

with  $\rho^\pm$  satisfying

$$p \rho^- (1 - \rho^+) = q \rho^+ (1 - \rho^-), \quad \frac{\rho^+ + \rho^-}{2} = \bar{\rho}, \tag{2.9}$$

where we used (2.5) and (2.7). This implies that in terms of the macroscopic variable  $x = i/N$ , that the steady state density  $\bar{\rho}(x)$  is linear with a discontinuity at the battery

$$\bar{\rho}(x) = \rho^+ + (\rho^- - \rho^+)x, \quad \text{with } \bar{\rho}(0) = \rho^+, \bar{\rho}(1) = \rho^-. \tag{2.10}$$

The local equilibrium (2.8) can be justified [7] and one can prove that after space/time rescaling, the microscopic equations lead to a macroscopic description of the local density by the heat equation. The battery yields non-linear boundary conditions

$$\forall t > 0, x \in ]0, 1[, \quad \begin{cases} \partial_t \rho(t, x) = \partial_x^2 \rho(t, x), \\ p \rho(t, 1)(1 - \rho(t, 0)) = q \rho(t, 0)(1 - \rho(t, 1)), \\ \partial_x \rho(t, 0) = \partial_x \rho(t, 1), \end{cases} \tag{2.11}$$

where  $\rho(t, x)$  stands for the local density at the macroscopic time  $t$  and position  $x$ .

### 2.2 The Case of a Slowly Varying Field

In contrast to the battery where the driving force is localized on a single bond, we will now consider the case of a smoothly varying macroscopic field  $E(x)$  along the ring. The microscopic model is now the weakly asymmetric simple exclusion process of  $M$  particles on a ring of  $N$  sites. A particle on site  $i$  jumps to its neighboring site on its right at rate  $p_i$

and to its neighboring site on its left at rate  $q_i$  provided that the target site is empty. By weak asymmetry, we mean that  $p_i$  and  $q_i$  have the following scaling dependence on the system size  $N$

$$p_i = 1 + \frac{1}{N} E\left(\frac{i}{N}\right), \quad q_i = 1 - \frac{1}{N} E\left(\frac{i}{N}\right).$$

When the integral of  $E(x)$  over the circle is non zero, the system will reach a non-equilibrium steady state for every fixed  $N$ .

The time evolution of the microscopic mean density is given by

$$\begin{aligned} \partial_\tau \mu_{N,\tau}(\eta_i) &= p_{i-1} \mu_{N,\tau}(\eta_{i-1}(1 - \eta_i)) - q_i \mu_{N,\tau}(\eta_i(1 - \eta_{i-1})) \\ &\quad - p_i \mu_{N,\tau}(\eta_i(1 - \eta_{i+1})) + q_{i+1} \mu_{N,\tau}(\eta_{i+1}(1 - \eta_i)), \end{aligned} \tag{2.12}$$

where  $\mu_{N,\tau}(\cdot)$  is the expectation value wrt the time evolving measure.

Assuming that, for large  $N$ , the density profile takes the following scaling form

$$\mu_{N,\tau}(\eta_i) = \rho\left(\frac{\tau}{N^2}, \frac{i}{N}\right), \tag{2.13}$$

and the correlation functions scale as

$$\mu_{N,\tau}(\eta_i \eta_j) - \mu_{N,\tau}(\eta_i) \mu_{N,\tau}(\eta_j) = \frac{1}{N} \mathcal{C}\left(\frac{\tau}{N^2}, \frac{i}{N}, \frac{j}{N}\right), \tag{2.14}$$

one can show that for  $x = i/N, t = \tau/N^2$

$$\begin{aligned} &p_{i-1} \mu_{N,\tau}(\eta_{i-1}(1 - \eta_i)) - q_i \mu_{N,\tau}(\eta_i(1 - \eta_{i-1})) \\ &\simeq \frac{1}{N} [-\partial_x \rho(t, x) + 2E(x)\rho(t, x)(1 - \rho(t, x))] + \frac{1}{N^2} W(t, x), \end{aligned}$$

where  $W$  depends on the two-point correlation function  $\mathcal{C}$  defined in (2.14). One then gets from (2.12) the macroscopic evolution equation

$$\partial_t \rho(t, x) = \partial_x [\partial_x \rho(t, x) - 2E(x)\rho(t, x)(1 - \rho(t, x))]. \tag{2.15}$$

This is the viscous Burgers equation on a ring  $x \in [0, 1]$ . A mathematical derivation of (2.15) can be found in [20, 28].

The macroscopic evolution of the battery model (2.11) can then be recovered by taking a large localized field  $E(x)$  at the origin with a given integral equal to  $K$ . According to (2.15) this leads, in the limit, to a jump of density across the origin

$$\log \frac{\rho^+}{1 - \rho^+} - \log \frac{\rho^-}{1 - \rho^-} = \int_{\rho^-}^{\rho^+} \frac{d\rho}{\rho(1 - \rho)} = 2K.$$

This expression is equivalent to (2.2) with  $2K = \log(p/q)$ . The effect of the battery should be understood as a force which maintains a fixed difference of chemical potentials across the origin.

### 2.3 The Zero Range Process and the Moving Battery Model

In Sect. 6.1, we will discuss the invariant measure of a ZRP driven by a battery. The microscopic dynamics is defined as follows. At site  $i$ , the occupation number  $\eta_i$  can take any integer value and a particle at site  $i$  performs a jump to the left or to the right with rate  $g(\eta_i)$ , where the function  $g$  is non-decreasing (to avoid the condensation phenomenon [17]) and  $g(0) = 0$ . The driving force is modeled by modifying the jump rates between sites 1 and  $N$ : a particle jumps from 1 to  $N$  with rate  $qg(\eta_1)$  and from  $N$  to 1 with rate  $pg(\eta_N)$ . The hydrodynamic limit for this model on  $\mathbb{Z}$  was derived in [21].

The ZRP driven by a battery is related to the SSEP on a ring driven by a “moving battery”, i.e. a special tagged particle with asymmetric jump rates  $p$  and  $q$ . One can think of this particle as being driven by an external field  $E$  with  $p/q = \exp(2E)$  [10, 18, 21]. A current will be induced by the moving battery. The mapping between the two models was exploited by [21] to study the motion of this tagged particle in  $\mathbb{Z}$ . More precisely, the SSEP with a moving battery and with  $M$  particles on the ring of length  $N$  can be mapped onto a ZRP on a ring of length  $M$  with  $N - M$  particles and with the specific rates  $g(n) = 1_{n \geq 1}$ . Let 1 be the index of the asymmetric particle in the SSEP and  $2, \dots, M$  the indices of the other particles, then the ZRP variable  $\eta_i$  stands for the number of empty spaces ahead of particle  $i$ .

## 3 The Macroscopic Approach for the Battery Model

A generic property of non-equilibrium systems, maintained in a steady state by contact with reservoirs at unequal chemical potentials, is the presence of long range correlations [4, 14, 16, 25, 27]. The steady state correlations can be predicted from a macroscopic approach based on “fluctuating hydrodynamics” [14, 24, 27]. Alternatively, one can derive the density large deviation functional for the steady state and then recover the correlations by expanding the functional near the steady state [1, 4, 8, 11]. We will apply the latter approach to compute the correlations in the battery model introduced in Sect. 2.1.

### 3.1 The Hydrodynamic Large Deviations

After rescaling the space by  $N$  ( $i = xN$ ) and the time by  $N^2$  ( $\tau = N^2t$ ), the microscopic system can be described at the hydrodynamic scale by two macroscopic functions the local density  $\rho(t, x)$  and the local current of particles  $q(t, x)$  which obey the conservation law

$$\partial_t \rho(t, x) = -\partial_x q(t, x). \tag{3.1}$$

The probability of observing a joint deviation of the current and the density in our microscopic system depends exponentially on the system size. More precisely, the probability of observing an atypical macroscopic trajectory  $(\rho(t, x), q(t, x))_{0 \leq t \leq T}$  during the macroscopic time interval  $[0, T]$  scales like

$$\mathbb{P}_{[0, TN^2]} \left( (\rho(t, x), q(t, x)) \right) \simeq \exp \left( -N \hat{\mathcal{F}}_{[0, T]}(\rho, q) \right),$$

where the large deviation functional is given by

$$\hat{\mathcal{F}}_{[0, T]}(\rho, q) = \int_0^T dt \int_0^1 dx \frac{(q(t, x) + \partial_x \rho(t, x))^2}{2\sigma(\rho(t, x))}. \tag{3.2}$$

Here  $\sigma(u) = 2u(1 - u)$  and the density has to satisfy the conditions

$$p\rho(t, 1)(1 - \rho(t, 0)) = q\rho(t, 0)(1 - \rho(t, 1)), \quad \int_0^1 dx \rho(t, x) = \bar{\rho}. \tag{3.3}$$

If  $(\rho(t, x), q(t, x))$  does not satisfy (3.1) or if the conditions (3.3) are not satisfied (for a set of times with non-zero Lebesgue measure) then the functional is infinite. The expression (3.2) is a generalization of the large deviation functionals derived for open systems (see [2, 6, 11] for reviews). The condition (3.3) at the battery follows from (2.2), as local equilibrium is still satisfied in the hydrodynamic large deviation regime.

To compute the probability of observing an atypical density trajectory  $(\rho(t, x))_{0 \leq t \leq T}$

$$\mathbb{P}_{[0, T]N^2_1}(\rho(t, x)) \simeq \exp(-N\mathcal{F}_{[0, T]}(\rho)), \quad \text{with } \mathcal{F}_{[0, T]}(\rho) = \inf_q \hat{\mathcal{F}}_{[0, T]}(\rho, q), \tag{3.4}$$

one has to optimize  $\hat{\mathcal{F}}_{[0, T]}(\rho, q)$  over all the currents which are compatible with the density (3.1) and which can be written as

$$q(t, x) = j(t) - \int_0^x \partial_t \rho(t, u) du, \tag{3.5}$$

where  $j(t)$  is the current at 0. Optimizing (3.2) over  $j(t)$  implies

$$\forall t > 0, \quad 0 = \int_0^1 dx \frac{j(t) - \int_0^x \partial_t \rho(t, u) du + \partial_x \rho(t, x)}{\sigma(\rho(t, x))}. \tag{3.6}$$

We introduce an auxiliary function  $H$  such that

$$q(t, x) = -\partial_x \rho(t, x) + \sigma(\rho(t, x))\partial_x H(t, x), \tag{3.7}$$

and get from (3.6)

$$0 = \int_0^1 dx \frac{q(t, x) + \partial_x \rho(t, x)}{\sigma(\rho(t, x))} = \int_0^1 dx \partial_x H(t, x) = H(t, 1) - H(t, 0).$$

This leads to a continuity condition on  $H$  at the battery

$$H(t, 0) = H(t, 1). \tag{3.8}$$

For a trajectory  $(\rho(t, x))_{0 \leq t \leq T}$ , there exists  $H$  satisfying (3.7) and such that

$$\partial_t \rho(t, x) = \partial_x^2 \rho(t, x) - \partial_x (\sigma(\rho(t, x))\partial_x H(t, x)), \tag{3.9}$$

thus the functional (3.4) reads

$$\mathcal{F}_{[0, T]}(\rho) = \frac{1}{2} \int_0^T dt \int_0^1 dx \sigma(\rho(t, x)) (\partial_x H(t, x))^2. \tag{3.10}$$

Note that finding the optimal trajectories  $(H, \rho)$  is equivalent to finding the optimal  $(q, \rho)$ .

### 3.2 Steady State Large Deviations

Following [1], the density large deviations for the steady state can be computed by using the hydrodynamic large deviations. For any smooth function  $\lambda(x)$  in  $[0, 1]$ , one defines

$$\mathcal{G}(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left\langle \exp \left( \sum_{i=1}^N \lambda \left( \frac{i}{N} \right) \eta_i \right) \right\rangle, \tag{3.11}$$

which is given according to [1] by

$$\mathcal{G}(\lambda) = \lim_{T \rightarrow \infty} \sup_{\rho} \left\{ \int_0^1 dx \lambda(x) \rho(0, x) - \mathcal{F}_{[-T, 0]}(\rho) \right\}, \tag{3.12}$$

where  $\mathcal{F}_{[-T, 0]}$  is given by (3.10). The supremum ranges over all the density trajectories  $\rho(t, x)$  in the macroscopic time interval  $[-T, 0]$  which are equal to the steady state  $\bar{\rho}(x)$  at time  $-T$ .

Given the function  $\lambda$  and a time  $-T$ , we look for the optimal density trajectory  $\rho$  in the variational problem (3.12). To determine this optimal trajectory, we consider a variation  $\rho \rightarrow \rho + \varphi$  and  $H \rightarrow H + h$ . From (3.9),  $\varphi$  and  $h$  satisfy

$$\partial_t \varphi = \partial_x^2 \varphi - \partial_x \left( \sigma'(\rho) H' \varphi + \sigma(\rho) h' \right). \tag{3.13}$$

To simplify notations, we have omitted the  $(t, x)$  dependence and used the shorthand  $H' = \partial_x H(t, x)$ .

The conditions (3.3) become

$$\frac{\varphi(t, 1)}{\rho(t, 1)(1 - \rho(t, 1))} = \frac{\varphi(t, 0)}{\rho(t, 0)(1 - \rho(t, 0))}, \quad \int_0^1 dx \varphi(t, x) = 0. \tag{3.14}$$

As the density is equal to  $\bar{\rho}$  at time  $-T$  then  $\varphi(-T, x) = 0$ .

For an optimal trajectory  $\rho$ , the first order perturbation in  $\varphi$  in the variational problem (3.12) should be equal to 0 so that

$$\int_0^1 dx \lambda(x) \varphi(0, x) - \int_{-T}^0 dt \int_0^1 dx \left( \frac{1}{2} \sigma'(\rho) (H')^2 \varphi + \sigma(\rho) H' h' \right) = 0. \tag{3.15}$$

Integrating by parts and using (3.13), one gets at time  $t$

$$\begin{aligned} & \int_0^1 dx \sigma(\rho) H' h' \\ &= - \int_0^1 dx \left( \partial_x (\sigma(\rho) h') \right) H + \left( H(t, 1) \sigma(\rho(t, 1)) h'(t, 1) - H(t, 0) \sigma(\rho(t, 0)) h'(t, 0) \right) \\ &= \int_0^1 dx \left( \partial_t \varphi - \partial_x^2 \varphi + \partial_x \left( \sigma'(\rho) H' \varphi \right) \right) H \\ & \quad + \left( H(t, 1) \sigma(\rho(t, 1)) h'(t, 1) - H(t, 0) \sigma(\rho(t, 0)) h'(t, 0) \right). \end{aligned}$$



Another integration by parts leads to

$$\begin{aligned} & \int_{-T}^0 dt \int_0^1 dx \sigma(\rho)H'h' \\ &= - \int_{-T}^0 dt \int_0^1 dx [\varphi \partial_t H - \varphi' H' + \varphi \sigma'(\rho)(H')^2] \\ & \quad + \int_0^1 dx \varphi(0, x)H(0, x) + \int_{-T}^0 dt [H(t, 1)j(t, 1) - H(t, 0)j(t, 0)], \end{aligned} \tag{3.16}$$

where  $j(t, x) = -\varphi'(t, x) + \sigma'(\rho(t, x))H'(t, x)\varphi(t, x) + \sigma(\rho(t, x))h'(t, x)$  is the current variation  $q \rightarrow q + j$ . The current  $j(t, x)$  is continuous at the battery (this can be checked by combining the condition  $\int_0^1 dx \varphi(t, x) = 0$  and (3.13) which says that  $\partial_t \varphi = -j'$ ). As  $H$  is also continuous at the battery (3.8), the last boundary term in (3.16) vanishes. One has

$$\begin{aligned} & \int_{-T}^0 dt \int_0^1 dx \sigma(\rho)H'h' \\ &= \int_{-T}^0 dt \int_0^1 dx [\varphi (-\partial_t H - \partial_x^2 H - \sigma'(\rho)(H')^2)] \\ & \quad + \int_0^1 dx \varphi(0, x)H(0, x) + \int_{-T}^0 dt [H'(t, 1)\varphi(t, 1) - H'(t, 0)\varphi(t, 0)]. \end{aligned}$$

Thus for any perturbation  $\varphi$ , the condition (3.15) can be rewritten

$$\begin{aligned} & \int_{-T}^0 dt \int_0^1 dx \left[ \varphi \left( \partial_t H + \partial_x^2 H + \frac{1}{2} \sigma'(\rho)(H')^2 \right) \right] + \int_0^1 dx (\lambda(x) - H(0, x))\varphi(0, x) \\ & \quad - \int_{-T}^0 dt [\sigma(\rho(t, 1))H'(t, 1) - \sigma(\rho(t, 0))H'(t, 0)] \frac{\varphi(t, 0)}{\sigma(\rho(t, 0))} = 0, \end{aligned} \tag{3.17}$$

where we used the identity (3.14).

Combining (3.17) and (3.9), the evolution equations for the optimal trajectory in (3.12) are

$$\forall t \in [-T, 0], \quad \begin{cases} \partial_t \rho = \partial_x^2 \rho - \partial_x(\sigma(\rho)H'), \\ \partial_t H = -\partial_x^2 H - \frac{1}{2}\sigma'(\rho)(H')^2, \end{cases} \tag{3.18}$$

with boundary conditions

$$\forall x \in [0, 1], \quad \rho(-T, x) = \bar{\rho}(x), \quad H(0, x) = \lambda(x). \tag{3.19}$$

We also have that at the battery, for any time  $t < 0$ ,

$$p\rho(t, 1)(1 - \rho(t, 0)) = q\rho(t, 0)(1 - \rho(t, 1)), \quad \rho'(t, 0) = \rho'(t, 1), \tag{3.20}$$

$$H(t, 0) = H(t, 1), \quad \rho(t, 1)(1 - \rho(t, 1))H'(t, 1) = \rho(t, 0)(1 - \rho(t, 0))H'(t, 0). \tag{3.21}$$

The boundary conditions follow from (3.3), (3.8) and (3.17). The continuity of  $\rho'$  in (3.20) is a consequence of the continuity of the current (3.7) at the battery and of (3.21). We stress the fact that  $H(0, x) = \lambda(x)$  in (3.19) may not satisfy the boundary conditions (3.21).

The bulk evolution equations (3.18) have already been derived in previous works [1, 8, 12], but the boundary conditions (3.20), (3.21) are specific to the battery model.

### 3.3 Linearized Equations

The two-point correlations in the steady state can be obtained by taking the second derivative of  $\mathcal{G}(\lambda)$  at  $\lambda = 0$  (see (3.11)) [1, 11]. This can be understood as follows. For a given  $N$ , one has at the second order in  $\lambda$

$$\begin{aligned} & \frac{1}{N} \log \left\langle \exp \left( \sum_{i=1}^N \lambda \left( \frac{i}{N} \right) \eta_i \right) \right\rangle \\ &= \frac{1}{N} \sum_{i=1}^N \left( \lambda \left( \frac{i}{N} \right) \langle \eta_i \rangle + \frac{1}{2} \lambda \left( \frac{i}{N} \right)^2 \langle \eta_i \rangle (1 - \langle \eta_i \rangle) \right) \\ & \quad + \frac{1}{2N} \sum_{i \neq j}^N \lambda \left( \frac{i}{N} \right) \lambda \left( \frac{j}{N} \right) \langle \eta_i; \eta_j \rangle + o(\lambda^2), \end{aligned} \tag{3.22}$$

where  $\langle \eta_i; \eta_j \rangle = \langle \eta_i \eta_j \rangle - \langle \eta_i \rangle \langle \eta_j \rangle$  is the connected two-point correlation function. The above expansion amounts to perturbing the system around the steady state  $\bar{\rho}(x)$  (2.10).

For  $N$  large, one expects from (3.11) and (3.22) that

$$\begin{aligned} \mathcal{G}(\lambda) &= \int_0^1 dx \bar{\rho}(x) \lambda(x) + \bar{\rho}(x) (1 - \bar{\rho}(x)) \frac{\lambda(x)^2}{2} \\ & \quad + \int_0^1 dx \int_x^1 dy C(x, y) \lambda(x) \lambda(y), \end{aligned} \tag{3.23}$$

where  $C(x, y)$  is the macroscopic non-equilibrium two-point correlation function

$$\langle \eta_i; \eta_j \rangle = \frac{1}{N} C \left( \frac{i}{N}, \frac{j}{N} \right), \tag{3.24}$$

at the leading order in  $N$ .  $C(x, y)$  will be computed in (3.37). We stress the fact that the equality (3.23) rests on the assumption that one can interchange the limits  $N \rightarrow \infty$  and  $\lambda \rightarrow 0$ .

Our goal now is to compute  $\mathcal{G}(\lambda)$  to the second order in  $\lambda$  from the variational problem (3.12) and to deduce from it  $C(x, y)$  (3.23). To any  $\lambda$ , one can associate an optimal density  $\rho_\lambda$  which minimizes the variational problem (3.12). By construction the density at time 0 satisfies  $\rho_\lambda(0, x) = \frac{\delta \mathcal{G}(\lambda)}{\delta \lambda}$  so that expanding  $\rho_\lambda(0, x)$  to the first order in  $\lambda$  will lead to the second derivative of  $\mathcal{G}(\lambda)$ . One can see from (3.23) that

$$\rho_\lambda(0, x) = \bar{\rho}(x) + \bar{\rho}(x) (1 - \bar{\rho}(x)) \lambda(x) + \int_0^1 dy C(x, y) \lambda(y). \tag{3.25}$$

To determine  $\rho_\lambda$  for small  $\lambda$ , we linearize the optimal evolution (3.18) around the steady state profile  $\bar{\rho}$  (2.10). For the sake of notation, we drop the subscript  $\lambda$  in  $\rho_\lambda$  and decompose

the trajectory as  $\rho_\lambda = \bar{\rho} + \varphi$  and  $H = h$ , where  $\varphi$  and  $h$  are small. At the first order, (3.18) becomes

$$\forall t \in [-T, 0], \quad \begin{cases} \partial_t \varphi = \partial_x^2 \varphi - \partial_x (\sigma(\bar{\rho})h'), \\ \partial_t h = -\partial_x^2 h, \end{cases} \tag{3.26}$$

and the boundary conditions (3.19), (3.20), (3.21) lead to

$$\forall x \in [0, 1], \quad \varphi(-T, x) = 0, \quad h(0, x) = \lambda(x), \tag{3.27}$$

and

$$\frac{\varphi(t, 1)}{\sigma(\rho^-)} = \frac{\varphi(t, 0)}{\sigma(\rho^+)}, \quad \varphi'(t, 0) = \varphi'(t, 1), \tag{3.28}$$

$$h(t, 1) = h(t, 0), \quad \sigma(\rho^-)h'(t, 1) = \sigma(\rho^+)h'(t, 0). \tag{3.29}$$

Note that (3.28) has been obtained by linearizing (3.20) and that  $h$  is defined up to a constant which does not influence the evolution of the density.

The coupled equations (3.26) can be solved by integrating the heat equation. We first start with the second equation. The Green’s function  $G^2$  associated to the Laplacian with boundary conditions (3.29) satisfies for any  $y$  in  $(0, 1)$

$$\partial_t G_t^2(x, y) = \partial_x^2 G_t^2(x, y), \quad \text{with } G_t^2(1, y) = G_t^2(0, y), \quad a \partial_x G_t^2(1, y) = \partial_x G_t^2(0, y), \tag{3.30}$$

with  $G_0^2(x, y) = \delta_{x=y}$  and

$$a = \frac{\sigma(\rho^-)}{\sigma(\rho^+)}. \tag{3.31}$$

The function  $h$  follows the backward heat equation on  $[-T, 0]$  with final data  $h(0, y) = \lambda(y)$  at time 0. Therefore, we can write

$$\forall t \in [-T, 0], \quad h(t, x) = \int_0^1 G_{-t}^2(x, y) \lambda(y) dy.$$

We turn now to the evolution of  $\varphi$  in (3.26) which follows the heat equation with a source term depending on  $h$ . The Green’s function  $G^1$  associated to the Laplacian with boundary conditions (3.28) satisfies for  $y$  in  $(0, 1)$

$$\partial_t G_t^1(x, y) = \partial_x^2 G_t^1(x, y), \quad \text{with } G_t^1(1, y) = a G_t^1(0, y), \quad \partial_x G_t^1(1, y) = \partial_x G_t^1(0, y), \tag{3.32}$$

with  $G_0^1(x, y) = \delta_{x=y}$ . Given  $\varphi(-T, y) = 0$  at time  $-T$

$$\forall t \in [-T, 0], \quad \varphi(t, x) = - \int_{-T}^t ds \int_0^1 dy G_{t-s}^1(x, y) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y h(s, y) \right).$$

One can check directly that the Green’s function solution of (3.32) is given by

$$G_t^1(x, y) = \frac{2}{a + 1}\alpha(x) + \frac{4}{a + 1} \sum_{k=1}^{\infty} \exp(- (2k\pi)^2 t) \left( \alpha(x) \cos(2\pi kx) \cos(2\pi ky) + \alpha(1 - y) \sin(2\pi kx) \sin(2\pi ky) + 4k\pi(1 - a)t \sin(2\pi kx) \cos(2\pi ky) \right), \tag{3.33}$$

with  $\alpha(x) = ax + 1 - x$  and  $a = \frac{\sigma(\rho^-)}{\sigma(\rho^+)}$ . One can also check that the Green’s function  $G^2$  which solves (3.30) is given by

$$G_t^2(x, y) = G_t^1(y, x). \tag{3.34}$$

In Appendix A, the construction of the Green’s functions is explained.

We are going to relate  $\varphi(0, x)$  and  $\lambda(x)$ . The density fluctuation at the final time reads

$$\varphi(0, x) = - \int_{-T}^0 ds \int_0^1 dy \int_0^1 dw G_{-s}^1(x, y) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y G_{-s}^2(y, w) \right) \lambda(w),$$

which is equivalent to

$$\varphi(0, x) = - \int_0^1 dw \left[ \int_0^T ds \int_0^1 dy G_s^1(x, y) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w) \right) \right] \lambda(w). \tag{3.35}$$

Let  $\mathcal{J} = \rho^+ - \rho^-$  be the rescaled mean current (for  $N$  large  $\mathcal{J} = N\hat{q}$  with  $\hat{q}$  introduced in (2.6)). In Appendix B, it is shown that (3.35) can be rewritten as

$$\varphi(0, x) = \frac{1}{2} \sigma(\bar{\rho}(x)) \lambda(x) + \int_0^1 dy C(x, y) \lambda(y), \tag{3.36}$$

where the long range correlations are given by

$$C(x, y) = - \frac{2}{(a + 1)^2} \left( \frac{\sigma(\rho^+) + \sigma(\rho^-)}{2} + \frac{\mathcal{J}^2}{3} \right) (ax + 1 - x)(ay + 1 - y) - 2\mathcal{J}^2 \int_0^{\infty} ds \left[ \int_0^1 dz G_s^1(x, z) G_s^1(y, z) - G_s^1(x, 0) G_s^1(y, 0) \right]. \tag{3.37}$$

Since  $\rho_\lambda(0, x) = \bar{\rho}(x) + \varphi(0, x)$ , this determines the correlation function in (3.25).

Note that when  $\mathcal{J} = 0$  then  $\rho^+ = \rho^-$ ,  $a = 1$  and the correlations reduce to  $C(x, y) = -\frac{\sigma(\bar{\rho})}{2}$  which are the correlations of the canonical measure (see also the discussion at the end of Sect. 5.1).

### 4 The Macroscopic Approach for the Slowly Varying Field

In Sect. 2.2, a microscopic dynamics driven by a weak field  $E(x)$  was introduced. We consider now a general diffusive system at a macroscopic level and write down the two-point correlation function which generalizes (3.37) to an arbitrary weak field  $E(x)$ . In the limit

where  $E(x)$  becomes a  $\delta$ -function, we will see that one can recover the correlations of the battery model (3.37).

The main advantage of the calculations of this section is that the effect of the battery is smoothened over the whole system through the function  $E(x)$ , so that there are no longer boundary conditions such as (3.20) or (3.21) at the battery and the functions  $H(t, x)$  and  $\rho(t, x)$  become simply periodic functions of the space variable  $x$ . One difficulty is that for a generic  $E(x)$  the steady state profile  $\bar{\rho}(x)$  and the Green's functions  $G^1$  and  $G^2$  are not known explicitly and we have to make a few assumptions on the convergence of hydrodynamic equation (4.5) to the steady state  $\bar{\rho}(x)$  or on the long time behavior of the Green's functions (4.18).

### 4.1 The Variational Problem

Our starting point is that the large deviation functional (3.2) of a macroscopic trajectory  $(\rho(t, x), q(t, x))_{0 \leq t \leq T}$  which, for a general diffusive system is characterized by two functions  $D(\rho)$  and  $\sigma(\rho)$ , takes the following form [3, 6] in presence of a driving field  $E$

$$\hat{\mathcal{F}}_{[0, T]}^E(\rho, q) = \int_0^T dt \int_0^1 dx \frac{(q(t, x) + D(\rho(t, x))\partial_x \rho(t, x) - E(x)\sigma(\rho(t, x)))^2}{2\sigma(\rho(t, x))}.$$

The SSEP with a weak field introduced in Sect. 2.2 corresponds to the functions  $D(\rho) = 1$  and  $\sigma(\rho) = 2\rho(1 - \rho)$ . Following exactly the same steps as in Sect. 3.1, one gets, by optimizing over the current  $j(t)$ , defined in (3.5), that (3.7) becomes

$$q(t, x) = -D(\rho(t, x))\partial_x \rho(t, x) + \sigma(\rho(t, x))[E(x) + \partial_x H(t, x)], \tag{4.1}$$

where  $H(t, x)$  is periodic in space ( $H(t, x) = H(t, x + 1)$ ). This implies that the time dependent density profile  $\rho$  is related to  $H$  by

$$\partial_t \rho(t, x) = \partial_x \left( D(\rho(t, x))\partial_x \rho(t, x) \right) - \partial_x \left( \sigma(\rho(t, x))[E(x) + \partial_x H(t, x)] \right), \tag{4.2}$$

(instead of (3.9)) and the density large deviation functional is given by

$$\mathcal{F}_{[0, T]}^E(\rho) = \frac{1}{2} \int_0^T dt \int_0^1 dx \sigma(\rho(t, x))(\partial_x H(t, x))^2, \tag{4.3}$$

as in (3.10).

To evaluate  $\mathcal{G}$  defined in (3.11), (3.12) we proceed as in Sect. 3.2 and determine the optimal dynamical fluctuation starting from the steady state  $\bar{\rho}(x)$  given as the solution of

$$-D(\bar{\rho}(x))\bar{\rho}'(x) + E(x)\sigma(\bar{\rho}(x)) = \mathcal{J}, \tag{4.4}$$

where  $\mathcal{J}$  is the steady state current. In Appendix C, we check that for regular coefficients  $\sigma(\rho), D(\rho) > 0$  and  $E(x)$  considered here (4.4) has a unique solution. We shall further assume that the hydrodynamic evolution

$$\partial_t \rho = \partial_x \left( D(\rho)\partial_x \rho - E(x)\sigma(\rho) \right), \tag{4.5}$$

converges to the steady state profile  $\bar{\rho}(x)$  (4.4).

As in Sect. 3.2, one considers a small variation  $\rho \rightarrow \rho + \varphi$ ,  $H \rightarrow H + h$  of  $\rho(t, x)$  and of  $H(t, x)$  and one gets from (4.2) that (3.13) becomes

$$\partial_t \varphi = \partial_x^2 (D(\rho)\varphi) - \partial_x (E(x)\sigma'(\rho)\varphi + \sigma'(\rho)H'\varphi + \sigma(\rho)h'). \tag{4.6}$$

Then following exactly the same steps as in Sect. 3.2, one ends up with (3.18) replaced by

$$\partial_t \rho = \partial_x (D(\rho)\partial_x \rho - E(x)\sigma(\rho) - \sigma(\rho)\partial_x H), \tag{4.7}$$

$$\partial_t H = -D(\rho)\partial_x^2 H - E(x)\sigma'(\rho)\partial_x H - \frac{1}{2}\sigma'(\rho)(\partial_x H)^2, \tag{4.8}$$

with the boundary conditions at times  $-T$  and 0 as in (3.19)

$$\forall x \in [0, 1], \quad \rho(-T, x) = \bar{\rho}(x), \quad H(0, x) = \lambda(x). \tag{4.9}$$

The spatial boundary conditions (3.20), (3.21) are now replaced by the requirement that  $H(t, x)$  and  $\rho(t, x)$  are smooth periodic functions of the space variable  $x$  (except possibly at time  $t = 0$ ).

#### 4.2 Small Variations of the Density

In order to determine the correlation function  $C_E(x, y)$  as in (3.25) we need to solve the above equations (4.7)–(4.9) to first order in  $\lambda(x)$ , that is to first order in  $\varphi = \rho - \bar{\rho}$  and  $h = H$ . Linearizing (4.7), (4.8), one gets

$$\partial_t \varphi = \partial_x ( \partial_x (D(\bar{\rho})\varphi) - E(x)\sigma'(\bar{\rho})\varphi - \sigma(\bar{\rho})\partial_x h ), \tag{4.10}$$

$$\partial_t h = -D(\bar{\rho})\partial_x^2 h - E(x)\sigma(\bar{\rho})'\partial_x h. \tag{4.11}$$

As in Sect. 3, these equations can be solved in two steps. As the equation for  $h$  does not involve  $\varphi$ , it is determined in terms of the Green’s function  $G^2$  as

$$\forall t \in [-T, 0], \quad h(t, x) = \int_0^1 G_{-t}^2(x, y)\lambda(y) dy, \tag{4.12}$$

where  $G^2$  is solution of

$$\partial_t G_t^2(x, y) = D(\bar{\rho}(x))\partial_x^2 G_t^2(x, y) + E(x)\sigma'(\bar{\rho}(x))\partial_x G_t^2(x, y), \tag{4.13}$$

with  $G_0^2(x, y) = \delta_{x=y}$ . To solve for  $\varphi$ , it is convenient to write  $\varphi$  as

$$\varphi(t, x) = \frac{\sigma(\bar{\rho}(x))}{2D(\bar{\rho}(x))}h(t, x) + \psi(t, x). \tag{4.14}$$

The evolution equation of  $\psi$  can be determined from (4.8)

$$\partial_t \psi = \partial_x^2 [D\psi] - \partial_x [E\sigma'\psi] + h\partial_x \left[ \frac{\sigma'\bar{\rho}'}{2} - \frac{E\sigma\sigma'}{2D} \right],$$

where the functions  $D, \sigma, \sigma'$  are evaluated at density  $\bar{\rho}(x)$  and  $E$  at position  $x$ . Using the fact that the steady state profile  $\bar{\rho}(x)$  satisfies (4.4), this becomes

$$\partial_t \psi = \partial_x^2 [D\psi] - \partial_x [E\sigma'\psi] - \mathcal{J}h\partial_x \left[ \frac{\sigma'}{2D} \right]. \tag{4.15}$$

We introduce the Green's function  $G_T^1$  as the solution of

$$\partial_t G_T^1(x, y) = \partial_x^2 \left( D(\bar{\rho}(x))G_T^1(x, y) \right) - \partial_x \left( E(x)\sigma'(\bar{\rho}(x))G_T^1(x, y) \right), \tag{4.16}$$

with  $G_0^1(x, y) = \delta_{x=y}$ . Then one can solve (4.15)

$$\begin{aligned} \varphi(x, 0) &= \frac{\sigma(\bar{\rho}(x))}{2D(\bar{\rho}(x))} \lambda(x) - \mathcal{J} \int_0^T dt \int dz \int dy G_T^1(x, z) \partial_z \left( \frac{\sigma'(\bar{\rho}(z))}{2D(\bar{\rho}(z))} \right) G_T^2(z, y) \lambda(y) \\ &\quad - \int dz \int dy G_T^1(x, z) \frac{\sigma(\bar{\rho}(z))}{2D(\bar{\rho}(z))} G_T^2(z, y) \lambda(y). \end{aligned} \tag{4.17}$$

We assume for all  $x$  the convergence when  $T \rightarrow \infty$

$$G_T^1(x, z) = G_T^2(z, x) \rightarrow B(x). \tag{4.18}$$

For the battery model  $B(x)$  has the explicit form  $\frac{2\alpha(x)}{\alpha+1}$  (see (3.33)). Taking the limit  $T \rightarrow \infty$  in (4.17) gives the correlation functions

$$\begin{aligned} C_E(x, y) &= -B(x)B(y) \int dz \frac{\sigma(\bar{\rho}(z))}{2D(\bar{\rho}(z))} \\ &\quad - \mathcal{J} \int_0^\infty dt \int dz G_t^1(x, z) \partial_z \left( \frac{\sigma'(\bar{\rho}(z))}{2D(\bar{\rho}(z))} \right) G_t^2(z, y). \end{aligned}$$

As for the battery model, the Green's functions satisfy at any time  $t$  the symmetry property (3.34) (see the proof at the end of this Sect. 4.2)

$$G_t^2(x, y) = G_t^1(y, x). \tag{4.19}$$

Thus we finally get the macroscopic expression for the two point correlation functions with a weak field

$$\begin{aligned} C_E(x, y) &= -B(x)B(y) \int dz \frac{\sigma(\bar{\rho}(z))}{2D(\bar{\rho}(z))} \\ &\quad - \mathcal{J} \int_0^\infty dt \int dz G_t^1(x, z) G_t^1(y, z) \partial_z \left( \frac{\sigma'(\bar{\rho}(z))}{2D(\bar{\rho}(z))} \right). \end{aligned} \tag{4.20}$$

*Remark 4.1* When  $E(x) = E$  is constant, the stationary profile  $\bar{\rho}$  is constant and the second term in (4.20) vanishes. Moreover  $B(x) = 1$  (from (4.16), (4.18)) and the fact that  $\int_0^1 dy G_t^1(x, y) = 1$  so that  $C_E(x, y) = -\frac{\sigma(\bar{\rho})}{2D(\bar{\rho})}$  as expected from [4].

We now sketch a proof of the symmetry (4.19) which follows from the fact that the operators associated to the evolutions (4.13), (4.16) are adjoint. A solution  $f(t, x)$  of

$$\partial_t f(t, x) = D(\bar{\rho}(x))\partial_x^2 f(t, x) + E(x)\sigma'(\bar{\rho}(x))\partial_x f(t, x)$$

with initial condition  $f(0, x)$ , is, by definition of  $G^2$  (4.13), equal to

$$f(t, x) = \int dy G_t^2(x, y) f(0, y).$$

Now using the fact that

$$f(t + dt, x) = \int dy G_{t+dt}^2(x, y) f(0, y) = \int dy G_t^2(x, y) f(dt, y),$$

one can show after an integration by parts that

$$\partial_t G^2 = \partial_y^2 [D(\bar{\rho}(y))G^2] - \partial_y [E(y)\sigma'(\bar{\rho}(y))G^2].$$

Therefore  $G_t^2(x, y)$  evolves according to the same equation as  $G_t^1(y, x)$  (4.16) and they coincide at time  $t = 0$  ( $G_0^2(x, y) = G_0^1(y, x) = \delta_{x=y}$ ). Thus one concludes that they are identical.

### 4.3 The Case of a Battery

To represent our model of a battery as a limit of the smooth continuum problem discussed above, we consider the field  $E(x)$  localized in a certain region of size  $\Delta x$  around the origin with  $E(x) = 0$  outside the region  $[1 - \frac{\Delta x}{2}, 1] \cup [0, \frac{\Delta x}{2}]$  (where 1 is identified to 0 on the ring). Then one should take the difference of the chemical potentials  $\mu(\frac{\Delta x}{2})$ ,  $\mu(1 - \frac{\Delta x}{2})$  at the edge of the battery to be fixed. Using the relation between the chemical potential and the density of systems in local equilibrium, one has

$$\mu\left(\frac{\Delta x}{2}\right) - \mu\left(1 - \frac{\Delta x}{2}\right) = \int_{\rho(t, 1 - \frac{\Delta x}{2})}^{\rho(t, \frac{\Delta x}{2})} \frac{2D(\rho)}{\sigma(\rho)} d\rho = K', \tag{4.21}$$

where the constant  $K'$  is a characteristic of the battery which remains fixed as  $\Delta x \rightarrow 0$ .

For  $K' > 0$ , the picture is that for  $\Delta x$  small, the density has an abrupt increase for  $x$  in the region where the field is localized and a slow decay for  $\frac{\Delta x}{2} < x < 1 - \frac{\Delta x}{2}$ . In the limit  $\Delta x \rightarrow 0$ , the steady state profile  $\bar{\rho}(x)$  satisfies  $-D(\bar{\rho}(x))\bar{\rho}'(x) = \mathcal{J}$  for  $0 < x < 1$  with a jump from  $\rho^- = \bar{\rho}(1)$  to  $\rho^+ = \bar{\rho}(0)$ . The current  $\mathcal{J}$  and the densities  $\rho^-$  and  $\rho^+$  at the edges of the battery then satisfy

$$\int_{\rho^-}^{\rho^+} \frac{2D(\rho)}{\sigma(\rho)} d\rho = K', \quad \int_0^1 \bar{\rho}(x) dx = \bar{\rho}, \quad \int_{\rho^-}^{\rho^+} D(\rho) d\rho = \mathcal{J}.$$

If we assume that in the limit  $\Delta x \rightarrow 0$ , the Green's function  $G_t^1(x, z)$  converges to  $\hat{G}_t^1(x, z)$  then expression (4.20) becomes

$$\begin{aligned} \hat{C}(x, y) &= -B(x)B(y) \int dz \frac{\sigma(\bar{\rho}(z))}{2D(\bar{\rho}(z))} \\ &\quad - \mathcal{J} \int_0^\infty dt \int_0^1 dz \hat{G}_t^1(x, z) \partial_z \left( \frac{\sigma'(\bar{\rho}(z))}{2D(\bar{\rho}(z))} \right) \hat{G}_t^1(y, z) \\ &\quad - \mathcal{J} \int_0^\infty dt \left[ \frac{\sigma'(\rho^+)}{2D(\rho^+)} - \frac{\sigma'(\rho^-)}{2D(\rho^-)} \right] \hat{G}_t^1(x, 0) \hat{G}_t^1(y, 0). \end{aligned} \tag{4.22}$$



It is then easy to check that this expression of the pair correlation functions, for a general diffusive system, reduces to the one of the battery model (3.37) when one takes  $D(\rho) = 1$  and  $\sigma(\rho) = 2\rho(1 - \rho)$ . For the latter choice of  $D$  and  $\sigma$ ,  $\hat{G}_t^1$  coincides with the Green's function (3.33).

### 5 Comparison with the Open System and with Numerical Simulations

#### 5.1 Comparison with the Open System

For open diffusive non-equilibrium systems, the steady state has long range correlations which scale like  $1/N$  for a one-dimensional chain of length  $N$ . The two-point correlation function of the SSEP in contact with reservoirs at densities  $\rho_a, \rho_b$  is given by [15, 27]

$$1 \leq i < j \leq N, \quad \langle \eta_i; \eta_j \rangle_{\text{open}} = \frac{1}{N} C_{\text{open}} \left( \frac{i}{N}, \frac{j}{N} \right),$$

with

$$x < y, \quad C_{\text{open}}(x, y) = -\mathcal{J}_{\text{open}}^2 x(1 - y), \tag{5.1}$$

where  $\mathcal{J}_{\text{open}} = \rho_a - \rho_b$  stands for the steady state current.

The SSEP on a ring driven by a battery can be interpreted as the canonical non-equilibrium counterpart of the SSEP driven by reservoirs. By tuning the mean density  $\bar{\rho}$  and the asymmetry  $(p, q)$ , the boundary conditions at the battery  $\rho^+, \rho^-$  (2.9) can be chosen to be equal to  $\rho_a, \rho_b$  so that both systems have the same steady state density profile and the same steady state currents  $\mathcal{J} = \mathcal{J}_{\text{open}}$ . In both models local equilibrium holds and implies that at the leading order both steady state statistics are given locally by a product Bernoulli measure with the same density. In this sense, equivalence of ensembles is also satisfied in non-equilibrium. In order to understand more precisely the interplay between the non-equilibrium correlations, the canonical constraint and the driving mechanism we are going to compare the correlation functions (3.37) and (5.1) (as well as (4.20)).

Expression (5.1) is related to the inverse of the Laplacian  $\Delta^{\text{Dir}}$  in  $[0, 1]$  with Dirichlet boundary conditions. Let  $G_s^{\text{Dir}} = \exp(-s\Delta^{\text{Dir}})$  be the corresponding Green's function, then we know from [4, 27] that

$$C_{\text{open}}(x, y) = -\mathcal{J}^2 (\Delta^{\text{Dir}})^{-1}(x, y). \tag{5.2}$$

The inverse of the one-dimensional Laplacian  $\Delta^{\text{Dir}}$  coincides with the expression in (5.1). The relation (5.2) can be rewritten in terms of the Green's function  $G_s^{\text{Dir}}$  as a time dependent integral

$$\begin{aligned} C_{\text{open}}(x, y) &= -2\mathcal{J}^2 \int_0^\infty ds G_{2s}^{\text{Dir}}(x, y) \\ &= -2\mathcal{J}^2 \int_0^\infty ds \int_0^1 dz G_s^{\text{Dir}}(x, z) G_s^{\text{Dir}}(y, z), \end{aligned} \tag{5.3}$$

where  $\mathcal{J} = \mathcal{J}_{\text{open}}$ . In the same way, the correlation function for the weak field (4.20) solves an equation which does not depend on time. Let  $L_x$  be the linearized operator introduced in (4.16)

$$L_x = \partial_x^2 \left( D(\bar{\rho}(x)) \cdot \right) - \partial_x \left( E(x) \sigma'(\bar{\rho}(x)) \cdot \right).$$

Then (4.20) is equivalent to solve the equation in the periodic domain  $[0, 1]^2$

$$(L_x + L_y)C_E(x, y) = -\mathcal{J}\partial_x\left(\frac{\sigma'(\bar{\rho}(x))}{2D(\bar{\rho}(x))}\right)\delta_{x=y}, \quad \text{with } \int_0^1 \int_0^1 dx dy C_E(x, y) = 0. \tag{5.4}$$

This is similar to equation (4.4) of [4].

The response to a small drive of the open and close systems is different. For a small current  $\mathcal{J}$ , when the mean density  $\bar{\rho} \neq 1/2$  then the long range correlation (3.37) is of order  $\mathcal{J}$ . Instead (5.1) scales like  $\mathcal{J}^2$ . To see this, we first note that in the battery model the mean density profile is linear and  $\int_0^1 dx \bar{\rho}(x) = \frac{\rho^- + \rho^+}{2} = \bar{\rho}$ . Thus

$$\rho^- = \bar{\rho} - \frac{\mathcal{J}}{2}, \quad \rho^+ = \bar{\rho} + \frac{\mathcal{J}}{2}, \tag{5.5}$$

where the mean current is given by  $\mathcal{J} = \rho^+ - \rho^-$ . Expanding (3.37) to the first order in  $\mathcal{J}$  leads to

$$C(x, y) = -\frac{\sigma(\bar{\rho})}{2} \left( 1 + \frac{\sigma'(\bar{\rho})}{\sigma(\bar{\rho})} \mathcal{J}(1 - x - y) \right), \tag{5.6}$$

where we used the fact that  $a = 1 - \frac{\sigma'(\bar{\rho})}{\sigma(\bar{\rho})} \mathcal{J}$  to first order in  $\mathcal{J}$ . We note that this coincides with the first order expansion of the correlations (6.2) for the canonical measure associated to the product measure. At the order  $\mathcal{J}^2$  the non equilibrium contributions appear and (3.37) no longer matches with (6.2).

### 5.2 The Battery Model at Density 1/2

An explicit expression of the two-point correlations can be obtained by plugging in (3.37) the expression (3.33) of  $G_s^1$ . For general mean density, this expression is rather complicated. However, several simplifications occur when the mean density is equal to 1/2 and in the rest of this section we will focus on this case.

At mean density equal to 1/2, the relation (5.5) implies that the densities at the battery satisfy  $\rho^- = 1 - \rho^+$  so that  $a = \frac{\sigma(\rho^-)}{\sigma(\rho^+)} = 1$  and the boundary conditions (3.30), (3.32) simplify. In particular  $G_t^2(x, y) = G_t^1(x, y)$  and the Green’s functions are associated to the Laplacian on the ring  $[0, 1]$

$$G_t^1(x, y) = 1 + 2 \sum_{k=1}^{\infty} \exp(- (2k\pi)^2 t) \cos(2\pi k(x - y)).$$

As  $G_t^1(y, z) = G_t^1(z, y)$ , one gets

$$\begin{aligned} & \int_0^{\infty} ds \int_0^1 dz G_s^1(x, z)G_s^1(z, y) - G_s^1(x, 0)G_s^1(y, 0) \\ &= \int_0^{\infty} ds G_{2s}^1(x, y) - G_s^1(x, 0)G_s^1(y, 0) \\ &= \sum_{k \geq 1} \frac{\cos(2\pi k(x - y))}{(2k\pi)^2} - 2 \sum_{k=1}^{\infty} \frac{\cos(2\pi kx) + \cos(2\pi ky)}{(2k\pi)^2} \\ & \quad - 4 \sum_{k, n \geq 1} \frac{\cos(2\pi nx) \cos(2\pi ky)}{(2\pi)^2(k^2 + n^2)}. \end{aligned}$$

From the identity

$$\forall x \in [0, 1], \quad 2 \sum_{k \geq 1} \frac{\cos(2\pi kx)}{(2k\pi)^2} = -\frac{1}{2}x(1-x) + \frac{1}{12}.$$

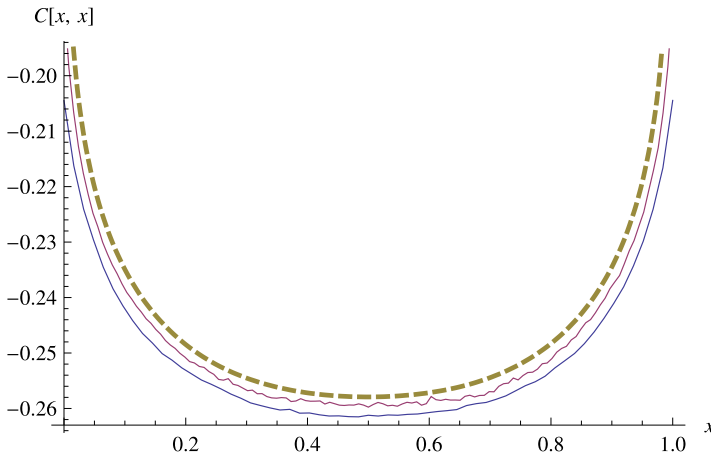
We deduce that for  $x < y$

$$\begin{aligned} & \int_0^\infty ds \int_0^1 dz G_s^1(x, z) G_s^2(z, y) - G_s^1(x, 0) G_s^1(y, 0) \\ &= \int_0^\infty ds G_{2s}^1(x, y) - G_s^1(x, 0) G_s^1(y, 0) \\ &= -\frac{1}{4}(y-x)(1+x-y) + \frac{1}{24} + \frac{1}{2}x(1-x) + \frac{1}{2}y(1-y) - \frac{1}{6} \\ &\quad - 4 \sum_{k, n \geq 1} \frac{\cos(2\pi nx) \cos(2\pi ky)}{(2\pi)^2(k^2 + n^2)}. \end{aligned}$$

Finally when the mean density is equal to 1/2 then (3.37) can be rewritten for  $x < y$

$$\begin{aligned} C(x, y) = & -\frac{1}{2}\sigma(\rho^+) + \frac{1}{12}\mathcal{J}^2 - \mathcal{J}^2 \left( \frac{1}{2}(x+y)(1-(x+y)) + x \right) \\ & + 8\mathcal{J}^2 \sum_{k, n \geq 1} \frac{\cos(2\pi nx) \cos(2\pi ky)}{(2\pi)^2(k^2 + n^2)}. \end{aligned} \tag{5.7}$$

We stress the fact that the correlation function diverges when both  $x, y$  approach the battery.



**Fig. 1** The *dashed line* represents the theoretical prediction (5.7) of  $C(x, x)$  at mean density  $\bar{\rho} = 1/2$ . The correlation  $N\langle \eta_i; \eta_{i+1} \rangle$  between the sites  $i$  and  $i + 1$  versus  $x = i/N$  is represented for systems of sizes 64 and 128 with  $25 \times 10^9$  updates per site. The results for  $L = 128$  lie between those of  $L = 64$  and the theoretical predictions

In Fig. 1, we compare the exact expression (5.7) to the results of a simulation, for  $N = 64$  and  $N = 128$  for  $25 \times 10^9$  updates per site. The agreement is very good. As our data for size  $N = 128$  lies between the data for  $N = 64$  and the theoretical prediction, it is reasonable to believe that the prediction does represent the  $N \rightarrow \infty$  limit.

## 6 Some Solvable Models

### 6.1 Zero Range Process

We study now the ZRP driven by a battery defined in Sect. 2.3. If  $p = q$  then the dynamics is reversible wrt the product measure  $\otimes_{i=1}^N \nu_\phi$ , where the measure at each site is given by

$$\nu_\phi(k) = \frac{1}{Z_\phi} \frac{\phi^k}{g(k)!}, \quad \text{with } g(k)! = \prod_{i=1}^k g(i),$$

where  $Z_\phi$  is the normalization factor and we used the convention  $g(0)! = 1$ .

Invariant measures of the ZRP have been determined for spatially varying jump rates in [17] (see (26)) and the battery model is a special case of these more general results. For the sake of completeness, we show below that for  $p \neq q$ , the invariant measure is a product measure with a canonical constraint.

For any  $\beta$  such that  $(q - p)\beta > 0$ , we introduce

$$\bar{\nu}_\beta = \otimes_{i=1}^N \nu_{\phi_i}, \quad \text{with } \phi_i = \beta \left( (i - 1) + \frac{p(N - 1) + 1}{q - p} \right). \tag{6.1}$$

Note that the measure  $\bar{\nu}_\beta$  is similar to the stationary measure of a ZRP in contact with reservoirs [23].

The invariant measure of the ZRP with a battery on a ring of length  $N$  with  $M$  particles is the measure  $\bar{\nu}_\beta$  (6.1) conditioned to a total number of particles equal to  $M$ . The conditional measure is independent of  $\beta$ .

In order to check that  $\bar{\nu}_\beta$  is invariant, we first introduce the generator of the zero range process

$$\begin{aligned} Lf(\eta) = & \sum_{i \neq 1, N} g(\eta_i) [(f(\eta^{(i, i+1)}) - f(\eta)) + (f(\eta^{(i, i-1)}) - f(\eta))] \\ & + g(\eta_1) [q(f(\eta^{(1, N)}) - f(\eta)) + (f(\eta^{(1, 2)}) - f(\eta))] \\ & + g(\eta_N) [p(f(\eta^{(N, 1)}) - f(\eta)) + (f(\eta^{(N, N-1)}) - f(\eta))], \end{aligned}$$

where  $\eta^{i,j}$  is the modified configuration after a jump from  $i$  to  $j$ . It is enough to show that for any function  $f$  then  $\bar{\nu}_\beta(Lf) = 0$ .

$$\begin{aligned} \bar{\nu}_\beta(Lf) = & \sum_{\eta} \bar{\nu}_\beta(\eta) \sum_{i \neq N} g(\eta_i) (f(\eta^{(i, i+1)}) - f(\eta)) + g(\eta_{i+1}) (f(\eta^{(i+1, i)}) - f(\eta)) \\ & + qg(\eta_1) (f(\eta^{(1, N)}) - f(\eta)) + pg(\eta_N) (f(\eta^{(N, 1)}) - f(\eta)) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\eta} \sum_{i \neq N} \bar{v}_{\beta}(\eta) f(\eta) \left[ g(\eta_{i+1}) \left( 1 + \frac{\phi_i - \phi_{i+1}}{\phi_{i+1}} \right) - g(\eta_{i+1}) \right. \\
 &\quad \left. + g(\eta_i) \left( 1 + \frac{\phi_{i+1} - \phi_i}{\phi_i} \right) - g(\eta_i) \right] \\
 &\quad + \bar{v}_{\beta}(\eta) f(\eta) \left[ qg(\eta_N) \left( 1 + \frac{\phi_1 - \phi_N}{\phi_N} \right) - qg(\eta_1) \right. \\
 &\quad \left. + pg(\eta_1) \left( 1 + \frac{\phi_N - \phi_1}{\phi_1} \right) - pg(\eta_N) \right] \\
 &= \sum_{\eta} \sum_{i \neq N} \bar{v}_{\beta}(\eta) f(\eta) (\phi_i - \phi_{i+1}) \left[ \frac{g(\eta_{i+1})}{\phi_{i+1}} - \frac{g(\eta_i)}{\phi_i} \right] \\
 &\quad + \bar{v}_{\beta}(\eta) f(\eta) \left[ g(\eta_N) \left( q - p + q \frac{\phi_1 - \phi_N}{\phi_N} \right) + g(\eta_1) \left( p - q + p \frac{\phi_N - \phi_1}{\phi_1} \right) \right].
 \end{aligned}$$

Since  $\phi_i$  varies linearly according to (6.1), we obtain

$$\begin{aligned}
 \bar{v}_{\beta}(Lf) &= \bar{v}_{\beta} \left( f(\eta) \left( \beta \left[ \frac{g(\eta_1)}{\phi_1} - \frac{g(\eta_N)}{\phi_N} \right] + g(\eta_N) \left( \frac{q\phi_1 - p\phi_N}{\phi_N} \right) \right. \right. \\
 &\quad \left. \left. + g(\eta_1) \left( \frac{p\phi_N - q\phi_1}{\phi_1} \right) \right) \right) = 0.
 \end{aligned}$$

The last equality follows from (6.1) which implies that  $q\phi_1 - p\phi_N = \beta$ .

The number of particles being preserved by the dynamics, the invariant measure is obtained by conditioning  $\bar{v}_{\beta}$  to have  $M$  particles.

The macroscopic approach is consistent with the expression of the invariant measure. The diffusion and conductivity coefficients of the ZRP satisfy  $D(\rho) = \sigma'(\rho)/2$ . Therefore in the case of a slowly varying weak field, the non-equilibrium part of the correlations  $C_E$  in (4.20) vanishes and only the canonical constraint remains.

### 6.2 Independent Variables and Canonical Constraints

We are going to compute the scaling limit of the truncated two-point correlations in product measures with canonical constraints. By analogy with the measures found in the previous section, we consider a chain of  $N$  independent variables  $\{\eta_i\}_{0 \leq i \leq N}$  with densities  $\rho_i = \bar{\rho}(i/N)$  which vary slowly from site to site.

Let  $\{v_{\rho}\}_{\rho}$  be a family of measures with different chemical potentials which are indexed by their density  $\rho$ . The variance of the measure  $v_{\rho}$  will be denoted by  $\chi(\rho)$ . We consider the measure  $\bar{\mu}$  which is the product measure  $\bigotimes_{i=1}^N v_{\rho_i}$  conditioned to have a total number of particles equal to the integer part of  $\sum_{i=1}^N \bar{\rho}(i/N)$ . The density profile of the measure  $\bar{\mu}$  is given by  $\bar{\rho}(x)$  and the two-point correlation function scales for large  $N$  and  $x \neq y$  in  $[0, 1]$  as

$$\bar{\mu}(\eta_{Nx}; \eta_{Ny}) = -\frac{1}{N} \frac{\chi(\bar{\rho}(x))\chi(\bar{\rho}(y))}{\int_0^1 \chi(\bar{\rho}(r)) dr}. \tag{6.2}$$

These asymptotics are very different from the long range non-equilibrium correlations of the SSEP with a battery and the same density profile (3.37).

To derive (6.2), one can approximate the original system in the large  $N$  limit by a chain of Gaussian independent variables under the canonical constraint. Define  $\mu$  to be the Gaussian measure of  $N$  independent Gaussian variables  $\{X_i\}_{1 \leq i \leq N}$  with variance  $\chi_i = \chi(\bar{\rho}(i/N))$  at site  $i$  under the canonical constraint  $\sum_{i=1}^N X_i = 0$ .

To compute the marginal of  $\mu$  over the two variables  $\{X_1, X_2\}$ , we integrate over the  $N - 2$  remaining Gaussian variables. Given  $\{X_1, X_2\}$  the probability density that  $\sum_{i=3}^N X_i = -X_1 - X_2$  is proportional to  $\exp(-\frac{1}{2\hat{\chi}}(X_1 + X_2)^2)$  with  $\hat{\chi} = \sum_{i=3}^N \chi_i$ . Thus the marginal of  $\mu$  over the two variables  $\{X_1, X_2\}$  is Gaussian with correlation matrix

$$\begin{pmatrix} \frac{1}{\chi_1} + \frac{1}{\hat{\chi}} & \frac{1}{\hat{\chi}} \\ \frac{1}{\hat{\chi}} & \frac{1}{\chi_2} + \frac{1}{\hat{\chi}} \end{pmatrix}^{-1} = \frac{1}{\hat{\chi} + \chi_1 + \chi_2} \begin{pmatrix} \chi_1(\hat{\chi} + \chi_2) & -\chi_1\chi_2 \\ -\chi_1\chi_2 & \chi_2(\hat{\chi} + \chi_1) \end{pmatrix}.$$

For large  $N$ ,  $\hat{\chi} \simeq N \int_0^1 \chi(\bar{\rho}(r)) dr$ , so that the two point correlation scales like

$$\mu(X_1; X_2) \simeq -\frac{1}{N} \frac{\chi_1\chi_2}{\int_0^1 \chi(\bar{\rho}(r)) dr}.$$

Thus (6.2) follows.

### 7 Conclusion

In this paper, we studied several NESS with a canonical constraint on the particle number and computed the two-point correlation functions by using a macroscopic approach. We considered two driving mechanisms: either a microscopic strong field localized on one bond (the battery model), or a weak field with an intensity which varies smoothly on the macroscopic scale. The battery model can be recovered as a singular limit of the smoothly varying field. Different driving mechanisms lead to different stationary measures with different long range correlations. In fact, the long range correlations of the canonical models differ also from the correlations in systems maintained out of equilibrium by reservoirs.

**Acknowledgements** We thank P.K. Mohanty, S. Olla and E.R. Speer for helpful discussions. JL acknowledges the support of the NSF Grant DMR-044-2066 and AFOSR Grant AF-FA9550-04. BD and TB acknowledge the support of the French Ministry of Education through the ANR BLAN07-2184264 grant. The work of TB was partially supported by the NSF Grant DMR-044-2066 and AFOSR Grant AF-FA9550-04 during a stay at Rutgers University and by the *Florence Gould Foundation Endowment* during a stay at the Institute for Advanced Study.

### Appendix A: Green’s Functions

In this appendix, we derive the exact expressions (3.33), (3.34) for the Green’s functions  $G^1, G^2$  which satisfy (3.30), (3.32). We recall that  $a = \frac{\sigma(\rho^-)}{\sigma(\rho^+)}$ .

We start with the computation of  $G_t^1$  introduced in (3.32) which evolves according to

$$\partial_t G_t^1(x, y) = \partial_x^2 G_t^1(x, y), \quad \text{with } G_t^1(1, y) = aG_t^1(0, y), \quad \partial_x G_t^1(1, y) = \partial_x G_t^1(0, y), \tag{A.1}$$

with  $G_0^1(x, y) = \delta_{x=y}$ . The Laplacian with boundary conditions (A.1) at the battery is not self-adjoint, but it can be decomposed on the following basis. We define

$$f_k^1(x) = (ax + 1 - x) \cos(2k\pi x), \quad f_k^2(x) = \sin(2k\pi x). \tag{A.2}$$

These functions satisfy the boundary conditions (A.1). Furthermore

$$\partial_x^2 f_k^1(x) = -(2k\pi)^2 f_k^1(x) + 4k\pi(1 - a) f_k^2(x), \quad \partial_x^2 f_k^2(x) = -(2k\pi)^2 f_k^2(x).$$

Let  $f_k^1(t, x), f_k^2(t, x)$  be the solution at time  $t$  of the heat equation with boundary conditions (3.28) and initial data  $f_k^1(x), f_k^2(x)$ , then

$$\begin{cases} f_k^1(t, x) = \exp(-(2k\pi)^2 t)(f_k^1(x) + 4k\pi(1 - a)t f_k^2(x)), \\ f_k^2(t, x) = \exp(-(2k\pi)^2 t) f_k^2(x). \end{cases} \tag{A.3}$$

The Green’s function  $G^2$  introduced in (3.30) satisfies

$$\partial_t G_t^2(x, y) = \partial_x^2 G_t^2(x, y), \quad \text{with } G_t^2(1, y) = G_t^2(0, y), \quad a \partial_x G_t^2(1, y) = \partial_x G_t^2(0, y), \tag{A.4}$$

with  $G_0^2(x, y) = \delta_{x=y}$ . For the Laplacian with boundary conditions (A.4), we can introduce the basis

$$g_k^1(x) = \cos(2k\pi x), \quad g_k^2(x) = (a + x - ax) \sin(2k\pi x), \tag{A.5}$$

and one has

$$\partial_x^2 g_k^1(x) = -(2k\pi)^2 g_k^1(x), \quad \partial_x^2 g_k^2(x) = -(2k\pi)^2 g_k^2(x) + 4k\pi(1 - a)g_k^1(x).$$

If  $g_k^1(t, x), g_k^2(t, x)$  denote the solution at time  $t$  of the heat equation with boundary conditions (A.4) and initial data  $g_k^1(x), g_k^2(x)$ , then

$$\begin{cases} g_k^1(t, x) = \exp(-(2k\pi)^2 t) g_k^1(x), \\ g_k^2(t, x) = \exp(-(2k\pi)^2 t) (4k\pi(1 - a)t g_k^1(x) + g_k^2(x)). \end{cases} \tag{A.6}$$

Using the fact that for  $x, y \in (0, 1)$

$$\sum_{n=0}^{\infty} \cos(2n\pi(x - y)) = \frac{1}{2} + \frac{1}{2} \delta_{x=y},$$

one can check that

$$\begin{aligned} & \frac{ay + 1 - y}{2} + \frac{a + 1}{4} \delta_{x=y} \\ &= \sum_{k=0}^{\infty} (ay + 1 - y) \cos(2k\pi y) \cos(2k\pi x) + (a + x - ax) \sin(2k\pi y) \sin(2k\pi x) \\ &= \sum_{k=0}^{\infty} f_k^1(y) g_k^1(x) + f_k^2(y) g_k^2(x). \end{aligned}$$

This implies

$$\frac{a + 1}{4} \delta_{x=y} = \frac{1}{2} f_0^1(y) g_0^1(x) + \sum_{k=1}^{\infty} f_k^1(y) g_k^1(x) + f_k^2(y) g_k^2(x). \tag{A.7}$$

Thus the Green’s function  $G^1$  is given by

$$\begin{aligned} \frac{a + 1}{4} G_t^1(x, y) &= \frac{1}{2} f_0^1(x) + \sum_{k=1}^{\infty} f_k^1(t, x) g_k^1(y) + f_k^2(t, x) g_k^2(y) \\ &= \frac{1}{2} f_0^1(x) + \sum_{k=1}^{\infty} \exp(- (2k\pi)^2 t) \\ &\quad \times \left( f_k^1(x) g_k^1(y) + f_k^2(x) g_k^2(y) + 4k\pi(1 - a) t f_k^2(x) g_k^1(y) \right). \end{aligned} \tag{A.8}$$

Similarly by exchanging  $x$  and  $y$  in the r.h.s. of (A.7) one can see that

$$\begin{aligned} \frac{a + 1}{4} G_t^2(x, y) &= \frac{1}{2} f_0^1(y) + \sum_{k=1}^{\infty} \exp(- (2k\pi)^2 t) \\ &\quad \times \left( f_k^1(y) g_k^1(x) + f_k^2(y) g_k^2(x) + 4k\pi(1 - a) t f_k^2(y) g_k^1(x) \right). \end{aligned} \tag{A.9}$$

This proves the expression (3.33) and the identity  $G_t^2(x, y) = G_t^1(y, x)$  claimed in (3.34).

### Appendix B

In this appendix, we are going to derive the relation (3.25) from (3.35).

We first start by proving some identities. The correlation between  $x$  and  $w$  is symmetric so that for any  $x$  and  $w$

$$\int_0^1 dy G_s^1(x, y) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w) \right) = \int_0^1 dy G_s^1(w, y) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y G_s^2(y, x) \right).$$

Using the fact that  $G_s^1(x, y) = G_s^2(y, x)$  (3.34), one gets

$$\begin{aligned} &\int_0^1 dy G_s^1(x, y) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w) \right) \\ &= \int_0^1 dy G_s^2(y, w) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y G_s^1(x, y) \right). \end{aligned} \tag{B.1}$$

Expanding each terms of (B.1) leads to the two identities,

$$\begin{aligned} &\int_0^1 dy G_s^1(x, y) \partial_y \left( \sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w) \right) \\ &= \int_0^1 dy G_s^1(x, y) \partial_y \sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w) + \int_0^1 dy G_s^1(x, y) \sigma(\bar{\rho}(y)) \partial_y^2 G_s^2(y, w) \end{aligned}$$



$$= \int_0^1 dy \partial_y G_s^1(x, y) \partial_y \sigma(\bar{\rho}(y)) G_s^2(y, w) + \int_0^1 dy \partial_y^2 G_s^1(x, y) \sigma(\bar{\rho}(y)) G_s^2(y, w). \tag{B.2}$$

We introduce the notation  $[F(y)]_{y=0}^{y=1} = F(1) - F(0)$ . Integrating by parts the first term in the last equation of (B.2), one gets

$$\begin{aligned} & \int_0^1 dy \partial_y G_s^1(x, y) \partial_y \sigma(\bar{\rho}(y)) G_s^2(y, w) \\ &= - \int_0^1 dy G_s^1(x, y) \partial_y (\partial_y \sigma(\bar{\rho}(y)) G_s^2(y, w)) + [G_s^1(x, y) \partial_y \sigma(\bar{\rho}(y)) G_s^2(y, w)]_{y=0}^{y=1} \\ &= - \int_0^1 dy G_s^1(x, y) \partial_y \sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w) - \int_0^1 dy G_s^1(x, y) \partial_y^2 \sigma(\bar{\rho}(y)) G_s^2(y, w) \\ & \quad + [G_s^1(x, y) \partial_y \sigma(\bar{\rho}(y)) G_s^2(y, w)]_{y=0}^{y=1}. \end{aligned} \tag{B.3}$$

By summing the two equalities in (B.2) and simplifying thanks to the identity (B.3), we finally obtain

$$\begin{aligned} & 2 \int_0^1 dy G_s^1(x, y) \partial_y (\sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w)) \\ &= \int_0^1 dy \partial_y^2 G_s^1(x, y) \sigma(\bar{\rho}(y)) G_s^2(y, w) + \int_0^1 dy G_s^1(x, y) \sigma(\bar{\rho}(y)) \partial_y^2 G_s^2(y, w) \\ & \quad - \int_0^1 dy G_s^1(x, y) \partial_y^2 \sigma(\bar{\rho}(y)) G_s^2(y, w) + [G_s^1(x, y) \partial_y \sigma(\bar{\rho}(y)) G_s^2(y, w)]_{y=0}^{y=1}. \end{aligned} \tag{B.4}$$

Note that  $\partial_y^2 \sigma(\bar{\rho}(y)) = -4\mathcal{J}^2$ , where  $\mathcal{J} = \rho^+ - \rho^-$  is the macroscopic mean current (for  $N$  large  $\mathcal{J} = N\hat{q}$  with  $\hat{q}$  introduced in (2.6)). The discontinuity of the density at the battery implies  $\partial_y \sigma(\rho^-) - \partial_y \sigma(\rho^+) = -4(\rho^+ - \rho^-)\mathcal{J} = -4\mathcal{J}^2$ . Thus integrating (B.4) wrt time leads to

$$\begin{aligned} & \int_0^T ds \int_0^1 dy G_s^1(x, y) \partial_y (\sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w)) \\ &= \int_0^1 dy G_T^1(x, y) \frac{\sigma(\bar{\rho}(y))}{2} G_T^2(y, w) - \frac{1}{2} \sigma(\bar{\rho}(x)) \delta_{x=w} \\ & \quad + 2\mathcal{J}^2 \int_0^T ds \int_0^1 dy G_s^1(x, y) G_s^2(y, w) - 2\mathcal{J}^2 \int_0^T ds G_s^1(x, 0) G_s^2(0, w). \end{aligned} \tag{B.5}$$

Letting  $T$  go to infinity, one gets from the expression of  $G^1$  (3.33) and  $G^2$  (3.34)

$$\lim_{T \rightarrow \infty} \int_0^1 dy G_T^1(x, y) \frac{\sigma(\bar{\rho}(y))}{2} G_T^2(y, w) = \frac{2}{(a+1)^2} \alpha(x) \alpha(w) \int_0^1 dy \sigma(\bar{\rho}(y)),$$

with the notation  $a = \frac{\sigma(\rho^-)}{\sigma(\rho^+)}$  and  $\alpha(x) = (ax + 1 - x)$ .

As the steady state is linear,  $\sigma(\bar{\rho}(y)) = \sigma(\rho^+) + y(\sigma(\rho^-) - \sigma(\rho^+) + 2\mathcal{J}^2) - 2\mathcal{J}^2y^2$ , and we get

$$\lim_{T \rightarrow \infty} \int_0^1 dy G_T^1(x, y) \frac{\sigma(\bar{\rho}(y))}{2} G_T^2(y, w) = \frac{2}{(a+1)^2} \alpha(x)\alpha(w) \left( \frac{\sigma(\rho^+) + \sigma(\rho^-)}{2} + \frac{\mathcal{J}^2}{3} \right).$$

Using (B.5) and the symmetry  $G_s^1(w, z) = G_s^2(z, w)$  (3.34), we have finally shown that

$$\lim_{T \rightarrow \infty} \int_0^T ds \int_0^1 dy G_s^1(x, y) \partial_y (\sigma(\bar{\rho}(y)) \partial_y G_s^2(y, w)) = -\frac{1}{2} \sigma(\bar{\rho}(x)) \delta_{x=w} - C(x, w),$$

with

$$C(x, w) = -\frac{2}{(a+1)^2} \left( \frac{\sigma(\rho^+) + \sigma(\rho^-)}{2} + \frac{\mathcal{J}^2}{3} \right) (ax + 1 - x)(aw + 1 - w) - 2\mathcal{J}^2 \int_0^\infty ds \int_0^1 dz G_s^1(x, z) G_s^1(w, z) - G_s^1(x, 0) G_s^1(w, 0). \tag{B.6}$$

The previous expression combined to (3.35) leads to (3.36).

## Appendix C

### C.1 Uniqueness of the Steady State

We will show the uniqueness of the smooth solutions of (4.4)

$$\partial_x \left( D(\bar{\rho}(x)) \bar{\rho}'(x) - E(x) \sigma(\bar{\rho}(x)) \right) = 0, \tag{C.1}$$

for regular coefficients and  $D$  positive.

Suppose that  $\rho_1(x), \rho_2(x)$  are two solutions of (C.1) with the same mean density. Then there is  $\mathcal{J}_1 > \mathcal{J}_2$  such that

$$\begin{aligned} -D(\bar{\rho}_1(x)) \bar{\rho}'_1(x) + E(x) \sigma(\bar{\rho}_1(x)) &= \mathcal{J}_1, \\ -D(\bar{\rho}_2(x)) \bar{\rho}'_2(x) + E(x) \sigma(\bar{\rho}_2(x)) &= \mathcal{J}_2. \end{aligned}$$

If the profiles coincide at  $x_0$  then subtracting both equations, one has

$$D(\bar{\rho}_1(x_0)) [\bar{\rho}'_1(x_0) - \bar{\rho}'_2(x_0)] = \mathcal{J}_2 - \mathcal{J}_1 < 0.$$

This implies that when the two solutions cross then  $\bar{\rho}_1 > \bar{\rho}_2$  before the crossing and  $\bar{\rho}_1 < \bar{\rho}_2$  after the crossing, so that the solutions cannot cross more than once. But the profiles are smooth, periodic and have the same mean density, thus they have to cross an even number of times. This is a contradiction and therefore both profiles have to be equal ( $\mathcal{J}_1 = \mathcal{J}_2$ ).

### C.2 Convergence of the Linearized Evolution

We turn now to the convergence of the linearized evolution (4.16)

$$\partial_t f(t, x) = \partial_x^2 \left( D(\bar{\rho}(x)) f(t, x) \right) - \partial_x \left( E(x) \sigma'(\bar{\rho}(x)) f(t, x) \right), \tag{C.2}$$

with smooth initial data  $f(0, x)$  and mean  $\int_0^1 dx f(0, x) = 1$ . It is equivalent to consider the evolution

$$\partial_t f(t, x) = \partial_x \left( D(\bar{\rho}(x)) \partial_x f(t, x) \right) - \partial_x \left( \alpha(x) f(t, x) \right), \tag{C.3}$$

with  $\alpha(x) = -\partial_x D(\bar{\rho}(x)) + E(x)\sigma'(\bar{\rho}(x))$ . At any time  $t > 0$ ,  $f(t, x)$  can then be interpreted as the probability density of a particle evolving on the ring  $[0, 1]$  with a non homogeneous diffusion coefficient  $D(\bar{\rho}(x))$  and a drift  $\alpha(x)$ . As the particle evolves on a compact set, it will reach a stationary state when  $t$  goes to infinity. The limiting density for the particle position will be denoted by  $\bar{f}(x)$  and it is the unique solution with mean density  $\int_0^1 dx \bar{f}(x) = 1$  of

$$\partial_x \left( D(\bar{\rho}(x)) \partial_x \bar{f}(x) \right) - \partial_x \left( \alpha(x) \bar{f}(x) \right) = 0. \tag{C.4}$$

A way to understand the relaxation to the stationary state is to check that the relative entropy

$$S(t) = - \int_0^1 dx f(t, x) \log \left( \frac{f(t, x)}{\bar{f}(x)} \right), \tag{C.5}$$

is a Lyapunov function for the evolution (C.3). This is a general fact for Markov processes. Taking the time derivative one has

$$\partial_t S(t) = \int_0^1 dx \left( D(\bar{\rho}(x)) \partial_x f(t, x) - \alpha(x) f(t, x) \right) \partial_x \log \left( \frac{f(t, x)}{\bar{f}(x)} \right). \tag{C.6}$$

We now note that

$$\begin{aligned} \int_0^1 dx \alpha(x) f(t, x) \partial_x \log \left( \frac{f(t, x)}{\bar{f}(x)} \right) &= \int_0^1 dx \alpha(x) \bar{f}(x) \partial_x \left( \frac{f(t, x)}{\bar{f}(x)} \right) \\ &= \int_0^1 dx D(\bar{\rho}(x)) \partial_x \bar{f}(x) \partial_x \left( \frac{f(t, x)}{\bar{f}(x)} \right), \end{aligned}$$

where we used (C.4) in the last equality. Combined with (C.6), this leads to

$$\begin{aligned} \partial_t S(t) &= \int_0^1 dx D(\bar{\rho}(x)) \left( \frac{\bar{f}(x)}{f(t, x)} \partial_x f(t, x) - \partial_x \bar{f}(x) \right) \partial_x \left( \frac{f(t, x)}{\bar{f}(x)} \right) \\ &= 4 \int_0^1 dx D(\bar{\rho}(x)) \bar{f}(x) \left( \partial_x \sqrt{\frac{f(t, x)}{\bar{f}(x)}} \right)^2. \end{aligned} \tag{C.7}$$

Thus  $S(t)$  is a Lyapunov function. A quantitative estimate of the approach to equilibrium could then be obtained by using a log-Sobolev inequality.

**References**

1. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Macroscopic fluctuation theory for stationary non equilibrium states. *J. Stat. Phys.* **107**, 635–675 (2002)

2. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Stochastic interacting particle systems out of equilibrium. *J. Stat. Mech.* P07014 (2007)
3. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Non equilibrium current fluctuations in stochastic lattice gases. *J. Stat. Phys.* **123**, 237–276 (2006)
4. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Towards a nonequilibrium thermodynamics: a self-contained macroscopic description of driven diffusive systems. *J. Stat. Phys.* **135**(5–6), 857–872 (2009)
5. Bodineau, T., Derrida, B.: Current fluctuations in non-equilibrium diffusive systems: an additivity principle. *Phys. Rev. Lett.* **92**, 180601 (2004)
6. Bodineau, T., Derrida, B.: Cumulants and large deviations of the current in non-equilibrium steady states. *C. R. Phys.* **8**, 540–555 (2007)
7. Bodineau, T., Lebowitz, J.: Work in progress
8. Bodineau, T., Derrida, B., Lecomte, V., van Wijland, F.: Long range correlations and phase transitions in non-equilibrium diffusive systems. *J. Stat. Phys.* **133**(6), 1013–1031 (2008)
9. Burdzy, K., Pal, S., Swanson, J.: Crowding of Brownian spheres. Preprint [arXiv:1002.1057](https://arxiv.org/abs/1002.1057)
10. Burlatsky, S.F., Oshanin, G., Moreau, M., Reinhardt, W.P.: Motion of a driven tracer particle in a one-dimensional symmetric lattice gas. *Phys. Rev. E* **54**, 3165 (1996)
11. Derrida, B.: Non-equilibrium steady states: fluctuations and large deviations of the density and of the current. *J. Stat. Mech.* P07023 (2007)
12. Derrida, B., Gerschenfeld, A.: Current fluctuations in one dimensional diffusive systems with a step initial profile. *J. Stat. Phys.* 137(5–6) (2009). [0907.3294](https://arxiv.org/abs/0907.3294) [cond-mat]
13. Derrida, B., Lebowitz, J., Speer, E.R.: Large deviation of the density drofile in the steady state of the symmetric simple exclusion process. *J. Stat. Phys.* **107**, 599–634 (2002)
14. Derrida, B., Enaud, C., Landim, C., Olla, S.: Fluctuations in the weakly asymmetric exclusion process with open boundary conditions. *J. Stat. Phys.* **118**, 795–811 (2005)
15. Derrida, B., Lebowitz, J., Speer, E.R.: Entropy of open lattice systems. *J. Stat. Phys.* **126**, 1083–1108 (2007)
16. Dorfman, J.R., Kirkpatrick, T.R., Sengers, J.V.: Generic long-range correlations in molecular fluids. *Ann. Rev. Phys. Chem.* **45**, 213–239 (1994)
17. Evans, M.R.: Phase transitions in one-dimensional nonequilibrium systems. *Braz. J. Phys.* **30**(1), (2000)
18. Ferrari, P., Goldstein, S., Lebowitz, J.: Diffusion, mobility and the Einstein relation. *Statistical Physics and Dynamical Systems. Progr. Phys.*, vol. 10, pp. 405–441. Birkhäuser Boston, Boston (1985)
19. Hinsch, H., Frey, E.: Bulk-driven non-equilibrium phase transitions in a mesoscopic ring. *Phys. Rev. Lett.* **97**, 095701 (2006)
20. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. *Grundlehren der Mathematischen Wissenschaften*, vol. 320. Springer, Berlin (1999)
21. Landim, C., Olla, S., Volchan, S.B.: Driven tracer particle in one dimensional symmetric simple exclusion. *Commun. Math. Phys.* **192**, 287–307 (1998)
22. Liggett, T.: *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, vol. 324. Springer, Berlin (1999)
23. de Masi, A., Ferrari, P.: A remark on the hydrodynamics of the zero-range processes. *J. Stat. Phys.* **36**(1–2), 81–87 (1984)
24. Ortiz de Zarate, J.M., Sengers, J.V.: On the physical origin of long-ranged fluctuations in fluids in thermal nonequilibrium states. *J. Stat. Phys.* **115**, 1341–1359 (2004)
25. Schmitz, R., Cohen, E.G.D.: Fluctuations in a fluid under a stationary heat-flux. 1. General theory. *J. Stat. Phys.* **39**, 285–316 (1985)
26. Schütz, G.M.: Exactly solvable models for many-body systems far from equilibri. In: Domb, C., Lebowitz, J. (eds.) *Phase Transitions and Critical Phenomena*, vol. 19, pp. 1–251. Academic Press, London (2000)
27. Spohn, H.: Long range correlations for stochastic lattice gases in a non-equilibrium steady state. *J. Phys. A* **16**, 4275–4291 (1983)
28. Spohn, H.: *Large Scale Dynamics of Interacting Particles*. Springer, Berlin (1991)